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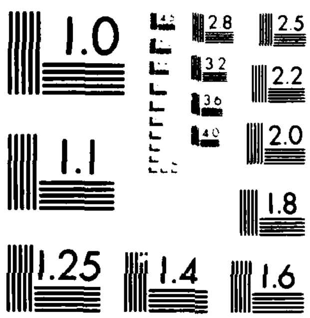

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AN APPROXIMATION FOR Z[G(s)F(s)] VIA THE SECOND MEAN VALUE THEOREM OF THE INTEGRAL CALCULUS: THE DIGITAL SIMULATION OF CONTINUOUS LINEAR SYSTEMS VIA Z-TRANSFORMS

Richard E. Dickson
Systems Simulation and Development Directorate
US Army Missile Laboratory

November 1982

U.S. ARMY MISSILE COMMAND
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This report presents the approximation for Z[G(s)F(s)], which is based upon the second mean value theorem of the integral calculus.

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Richard E. Dickson
Systems Simulation and Development Directorate
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Redstone Arsenal, AL 35898

Summary

The problem with applying z-transforms to the digital simulation of continuous linear systems is that the Raggazzini-Zadeh identity only applies to the initial conditions. This difficulty may be overcome by approximating Z[G(s)F(s)], for example, Halijak's "Trapezoidal Convolution"[1,2] and its extensions.[3] These approximations did not yield perfect response for a unit step into a single pole filter unless adjusted by a residue.[4]

The approximation for Z[G(s)F(s)] presented here, which is based upon the second mean value theorem of the integral calculus, has the desired response for a unit step into a single pole filter and does not require adjustment.

Some Preliminaries

The Laplace transform is defined to be

$$F(s) = \int_{-\infty}^{\infty} f(t)\, u(t)\, e^{-st}\, dt \qquad (1)$$

where $u(t)$ is the Heaviside unit step,

$$u(t) = \begin{cases} 0, t < 0 \\ 1, t \geq 0 \end{cases} \qquad (2)$$

The z-transform will be defined as a "discrete" Laplace transform

$$F(z) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(t)\, \delta(nT - t)\, u(t)\, e^{-st}\, dt \qquad (3)$$

and from the "sifting" property of the Dirac delta distribution,

$$f(nT) = \int_{-\infty}^{\infty} f(t)\, \delta(nT - t)\, dt \qquad (4)$$

equation (3) becomes

$$F(z) = \sum_{n=-\infty}^{\infty} f(nT)\, u(nT)\, e^{-snT} \qquad (5)$$

and letting

$$z = e^{sT} \qquad (6)$$

one has

$$F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n} \qquad (7)$$

the usual definition[2] of the z-transform.

The Raggazzini-Zadeh identity may be readily deduced from the convolution properties of the Laplace transform

$$Z[G(z)\, F(s)] = \sum_{n=-\infty}^{\infty} z^{-n}\, u(nT) \int_{\infty}^{\infty} \sum_{k=-\infty}^{\infty} g(t)\, \delta(kT - t)\, u(t)\, f(nT - t)\, u(nT - t)\, dt \qquad (8)$$

The "sifting" property of the Dirac delta, equation (4), yields

$$Z[G(z)F(s)] = \sum_{n=0}^{\infty} \sum_{k=0}^{n} g(kT) z^{-k}\, f(nT - kT) z^{-(n-k)} \qquad (9)$$

where the summation over k is discrete convolution. From the Cauchy product of power series

$$z\, G(z)F(s)] = \left(\sum_{k=0}^{\infty} g(kT) z^{-k} \right) \left(\sum_{n=0}^{\infty} f(nT) z^{-n} \right) \qquad (10)$$

and from equation (7)

$$Z[G(z)F(s)] = G(z)F(z) \qquad (11)$$

the Raggazzini-Zadeh identity.

Z[G(s)F(s)] Approximation

It is well known that, in general, $Z[G(s)F(s)] \neq T\, G(z)F(z)$ and therein lies the difficulty in applying z-transforms to the digital simulation of continuous systems. A very useful relationship would be the solution of

$$Z[G(s)F(s)] = \sum_{n=-\infty}^{\infty} z^{-n}\, u(nT) \qquad (12)$$
$$\times \int_{-\infty}^{\infty} g(t)\, u(t)\, f(nT - t)\, u(nT - t)\, dt$$

If both G(s) and F(s) are known *a priori*, there is no difficulty; but, in digital simulation, though the plant, G(s), is known *a priori*, the input, f(nT), is not.

By the second mean value theorem of the integral calculus, Bonnet's theorem, one may write

$$Z[G(s)F(s)] = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} \Big\{ f(nT - kT) \int_{kT}^{kT+\eta_k T} g(t)\, dt \qquad (13)$$
$$+ f(nT - kT - T) \int_{kT+\eta_k T}^{kT+T} g(t)\, dt \Big\} z^{-n}$$

In general η_k would be different for each k. Assume that η is the same for each k, that is,

$$\eta_k = \eta \qquad (14)$$

Equation (13) is almost, but not quite, the form of the Cauchy product of power series. To obtain the

proper form

$$Z[G(s)F(s)] \approx \sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n-1} f(nT-kT)\int_{kT}^{kT+\eta T} g(t)dt + \sum_{k=1}^{n} f(nT-kT)\int_{kT+(\eta-1)T}^{kT} g(t)\,dt\right\} z^{-n} \qquad (15)$$

$$\begin{aligned} Z[G(s)F(s)] \approx \sum_{n=0}^{\infty}\Bigg\{ & \sum_{k=0}^{n} f(nT-kT)z^{-(n-k)}\int_{kT}^{kT+\eta T} g(t)dt\, z^{-k} - f(0)\int_{nT}^{nT+\eta T} g(t)\,dt\, z^{-n} \\ & + \sum_{k=0}^{n} f(nT-kT)z^{-(n-k)}\int_{kT+(\eta-1)T}^{kT} g(t)dt\, z^{-k} - f(nT)\, z^{-n}\int_{(\eta-1)T}^{0} g(t)dt\Bigg\} \end{aligned} \qquad (16)$$

and finally

$$Z[G(s)F(s)] \approx [F(z)-f(o)]\sum_{n=0}^{\infty}\int_{nT}^{nT+\eta T} g(t)\,dt\, z^{-n} + F(z)\left[\sum_{n=0}^{\infty}\int_{nT+(\eta-1)T}^{nT} g(t)dt\, z^{-n} - \int_{(\eta-1)T}^{0} g(t)dt\right] \qquad (17)$$

It is interesting to compare this result with Halijak's "Trapezoidal Convolution:"[1,2]

$$Z[G(s)F(s)] \approx \frac{T}{2}\left\{[F(z)-f(o)]\,G(z) + F(z)\,[G(z)-g(o)]\right\} \qquad (18)$$

which may be derived from the first mean value theorem of the integral calculus.[3]

Two Plants of Interest

Consider the case where the plant is a single integrator; that is:

$$G(s) = 1/s \qquad (19a)$$

$$g(t) = 1 \qquad (19b)$$

Substituting into equation (18) one has

$$Z[F(s)/s] \approx [F(z)-f(o)]\sum_{n=0}^{\infty}\int_{nT}^{nT+\eta T} dt\, z^{-n} + F(z)\left[\sum_{n=0}^{\infty}\int_{nT+(\eta-1)T}^{nT} dt\, z^{-n} - \int_{(\eta-1)T}^{0} dt\right] \qquad (20)$$

and integrating

$$Z[F(s)/s] \approx [F(z)-f(o)]\sum_{n=0}^{\infty}\eta T\, z^{-n} + F(z)\left[\sum_{n=0}^{\infty}(1-\eta)T\, z^{-n} - (1-\eta)T\right] \qquad (21)$$

which may be written after summing

$$Z[F(s)/s] \approx [F(z)-f(o)]\,[\eta T/(1-z^{-1})] + F(z)\,[(1-\eta)T/(1-z^{-1}) - (1-\eta)T] \qquad (22)$$

and combining like terms

$$Z[F(s)/s] \approx \frac{[\eta + (1-\eta)z^{-1}]\,T\,F(z) - \eta T\, f(o)}{1-z^{-1}} \qquad (23)$$

For a unit step input,

$$F(s) = 1/s \qquad (24a)$$

that is

$$f(t) = 1 \qquad (24b)$$

and

$$F(z) = 1/(1-z^{-1}) \qquad (24c)$$

equation (23) becomes

$$Z[(1/s)/s] \approx \frac{[\eta + (1-\eta)z^{-1}]T}{(1-z^{-1})^2} - \frac{\eta T}{1-z^{-1}} \qquad (25)$$

$$Z[(1/s)/s] = \frac{T\,z^{-1}}{(1-z^{-1})^2} \qquad (26)$$

which is exact for any η.

For $F(s) = 1/s^2$ (27a)

that is $f(t) = t$ (27b)

and $F(z) = Tz/(1-z^{-1})^2$ (27c)

equation (23) becomes

$$Z[(1/s^2)/s] \approx \frac{[\eta + (1-\eta)z^{-1}]T}{1-z^{-1}}\left(\frac{T\,z^{-1}}{(1-z^{-1})^2}\right) \qquad (28)$$

$$Z[(1/s^2)/s] \approx \frac{[\eta + (1-\eta)z^{-1}]\,T^2 z^{-1}}{(1-z^{-1})^3} \qquad (29)$$

which for $\eta = 1/2$ is

$$Z[(1/s^2)/s] = \frac{T^2 z^{-1}(1+z^{-1})}{2(1-z^{-1})^3} \qquad (30)$$

which is exact.

A more interesting plant is the single pole filter:

$$G(s) = a/(s+a) \qquad (31a)$$

that is

$$g(t) = a\,e^{-aT} \tag{31b}$$

In this case equation (17) becomes

$$Z[a\,F(s)/(s+a)] \approx [F(z) - f(o)] \sum_{n=0}^{\infty} \int_{nT}^{nT+\eta T} ae^{-at}\,dt\, z^{-n} + F(z)\left[\sum_{n=0}^{\infty} \int_{nT+(\eta-1)T}^{nT} ae^{-at}dt\, z^{-n} - \int_{(\eta-1)T}^{0} ae^{-at}\,dt\right] \tag{32}$$

and after performing the indicated integration and sums

$$Z[a\,F(s)/(s+a)] \approx [F(z) - f(o)]\frac{\left(1 - e^{-a\eta T}\right)}{\left(1 - e^{-at}\,z^{-1}\right)} + F(z)\left[\frac{\left(e^{-a(\eta-1)T} - 1\right)e^{-at}\,z^{-1}}{1 - e^{-at}\,z^{-1}}\right] \tag{33}$$

In this case, "Trapezoidal Convolution," equation (18) would yield

$$Z[a\,F(s)/(s+a)] \approx \left(\frac{aT}{2}\right)\left\{[F(z) - f(o)] \,/\, (1-e^{-aT}\,z^{-1}) + F(z)\left[e^{-aT}\,z^{-1}/(1 - e^{-aT}\,z^{-1})\right]\right\} \tag{34}$$

and with residue adjustment

$$Z[a\,F(s)/(s+a)] \approx \tanh\left(\frac{aT}{2}\right)\left\{[F(z) - f(o)]/\left(1 - e^{-aT}\,z^{-1}\right) + F(z)\left[e^{-aT}\,z^{-1}/(1-e^{-aT}\,z^{-1})\right]\right\} \tag{35}$$

For a unit step input, $F(s) = 1/s$ (24a)

that is $f(t) = 1$ (24b)

and $F(z) = 1/(1 - z^{-1})$ (24c)

equation (33) becomes

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] \approx \left[\frac{1}{1 - z^{-1}} - 1\right]\left(\frac{1 - e^{-a\eta T}}{1 - e^{-aT}\,z^{-1}}\right) + \left(\frac{1}{1 - z^{-1}}\right)\left[\frac{\left(e^{-a\eta T} - e^{-aT}\right)z^{-1}}{1 - e^{-aT}\,z^{-1}}\right] \tag{36}$$

and collecting terms

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] = \frac{\left(1 - e^{-aT}\right)z^{-1}}{\left(1 - z^{-1}\right)\left(1 - e^{-aT}\,z^{-1}\right)} \tag{37}$$

which is exact for any η !

Without residue adjustment "trapezoidal convolution" yields[4]

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] \approx \left(\frac{aT}{2}\right)\frac{\left(1 + e^{-aT}\right)z^{-1}}{\left(1 - z^{-1}\right)\left(1 - e^{-aT}\,z^{-1}\right)} \tag{38}$$

With residue adjustment one has[4]

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] = \tanh\left(\frac{aT}{2}\right)\frac{\left(1 + e^{-aT}\right)z^{-1}}{\left(1 - z^{-1}\right)\left(1 - e^{-aT}\,z^{-1}\right)} \tag{39a}$$

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] = \frac{\left(1 - e^{-aT}\right)z^{-1}}{\left(1 - z^{-1}\right)\left(1 - e^{-aT}\,z^{-1}\right)} \tag{39b}$$

In the sample data domain a second order Runge-Kutta integrator would yield:

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] \approx \frac{\left[1 - (1 - aT + a^2T^2/2)\right]z^{-1}}{(1 - z^{-1})\left[1 - (1 - aT + a^2T^2/2)\,z^{-1}\right]} \tag{40}$$

Note the (2/0) Pade´ approximation for the exponential.

A unit step input and a Tustin's substitution[2] for the plant would lead to

$$Z\left[\frac{1}{s}\left(\frac{a}{s+a}\right)\right] \approx \frac{\left[1 - (1 - aT/2)/(1 + aT/2)\right]\;z^{-1}}{(1 - z^{-1})\left[1 - (1 - aT/2)z^{-1}/(1 + aT/2)\right]} \tag{41}$$

Note the (1/1) Pade´ approximations for the exponential.

For the large time step, T, the approximation presented here will out perform "Trapezoidal Convolution," Tustin's substitution method and a second order Runge-Kutta integrator.

"Trapezoidal Convolution"[1,2] yields exact results for a damped exponential input into a single real pole filter when the time constant of the input and the filter are the same[4]

$$Z\left[\frac{1}{s+a}\left(\frac{a}{s+a}\right)\right] = \frac{aT\,e^{-aT}\,z^{-1}}{\left(1 - e^{-aT}\,z^{-1}\right)^2} \tag{42}$$

In this case, equation (17) yields

$$Z\left[\frac{1}{s+a}\left(\frac{a}{s+a}\right)\right] \approx \frac{e^{-a\eta T}\left(1 - e^{-aT}\right)z^{-1}}{\left(1 - e^{-aT}\,z^{-1}\right)^2} \tag{43}$$

and setting this numerator equal to the exact numerator and solving for η , one has

$$\eta = \frac{-1}{aT}\ln\left(\frac{aT\,e^{-aT}}{1 - e^{-aT}}\right) \tag{44}$$

For $aT \ll 1$

$$\eta \approx 1/2 \tag{45}$$

Recurrence for A Single Pole Filter

To develop a recurrence for a single real pole filter with non-zero initial conditions, start with the differential equation

$$\dot{x}(t) + a\,x(t) = a\,f(t) \tag{46}$$

and transform to the Laplace domain using

$$L\left[f^{(n)}(t)\right] = s^nF(s) - \sum_{\ell=0}^{n-1} s^{n-\ell-1}f^{(\ell)}(0) \tag{47}$$

$$s\,X(s) - x(0) + a\,X(s) = a\,F(s) \tag{48}$$

solving for X(s) and then taking the z-transform,

$$X(z) = Z\left[\frac{aF(s)}{s+a}\right] + Z\left[\frac{x(0)}{s+a}\right] \tag{49}$$

Since the initial condition, $x(o)$, is a constant in the Laplace domain (an impulse in the time domain), the Raggazzini-Zadeh identity, Equation (11), would apply. The forcing function, $F(s)$, is another matter, and requires approximation. Using equation (33)

$$\left(1-z^{-1}e^{-aT}\right)X(z) = \left[\left(1-e^{-a\eta T}\right) + \left(e^{-a\eta T} - e^{-aT}\right)z^{-1}\right]F(z) - \left(1-e^{-a\eta T}\right)f(o) + x(o) \tag{50}$$

and substituting equation (7) and equating coefficients of like powers of "z"

$$x(0) = x(0) \tag{51a}$$

$$x(nT) \approx e^{-aT}\, x(nT-T) + \left(1 - e^{-a\eta T}\right) f(nT)$$
$$+ \left(e^{-a\eta T} - e^{-aT}\right) f(nT - T),\ n > 0 \qquad (51b)$$

For a second order damped filter with non-zero initial conditions[4]

$$X(s) = \frac{\omega_0^2\, F(s) + (s + 2\sigma)\, x(0) + \dot{x}(0)}{s^2 + 2\sigma s + \omega_0^2} \qquad (52)$$

where

$$0 < \frac{\sigma}{\omega_0} < 1$$

a partial fraction expansion yields

$$X(s) = \frac{1}{2\omega}\left[\frac{i}{s + \sigma + i\omega} + \frac{-i}{s + \sigma - i\omega}\right] \times$$
$$\left[\omega_0^2\, F(s) + \ x(0) + \dot{x}(0)\right]$$
$$+ \frac{1}{2}\left[\frac{1}{s + \sigma + i\omega} + \frac{1}{s + \sigma - i\omega}\right] x(0) \qquad (53)$$

where

$$\omega = \left(\omega_0^2 - \sigma^2\right)^{1/2} \qquad (54)$$

Let

$$X(s) = \frac{U(s) + V(s)}{2} \qquad (55)$$

where

$$U(s) = \frac{i\left[\omega_0^2\, F(s) + \sigma\, x(0) + \dot{x}(0)\right]/\omega + x(0)}{s + \sigma + i\omega} \qquad (56)$$

and

$$V(s) = \frac{-i\left[\omega_0^2\, F(s) + \sigma\, x(0) + \dot{x}(0)\right]/\omega + x(0)}{s + \sigma - i\omega} \qquad (57)$$

Since $V(s)$ is the complex conjugate of $U(s)$, only $u(nT)$ need be found. The problem becomes that of finding the recurrence for a single complex pole filter. In this case, equation (51 a,b) would become:

$$u(0) = x(0) + i\left[\sigma x(0) + \dot{x}(0)\right]/\omega \qquad (58a)$$

$$u(nT) = e^{-(\sigma+i\omega)T}\, u(nT - T) + i\omega\left(\frac{\omega_0}{\omega}\right)^2 \left[1 - e^{-\eta(\sigma+i\omega)T}\right] f(nT) + i\omega\left(\frac{\omega_0}{\omega}\right)^2 \left[e^{-\eta(\sigma+i\omega)T} - e^{-(\sigma+i\omega)T}\right] f(nT - T) \qquad (58b)$$

where

$$e^{-(\sigma+i\omega)T} = e^{-\sigma T}\,(\cos \omega T - i \sin \omega T) \qquad (59)$$

and

$$x(nT) = \text{Real}\ \ u(nT) \qquad (60a)$$

$$\dot{x}(nT) = \omega\ \text{Imaginary}\ [u(nT)] - \sigma\ \text{Real}[u(nT)] \qquad (60b)$$

Conclusions

The approximation proposed here is exact for a unit step input into a single pole filter. The pole may be either real, imaginary, or complex. The response to a unit step is unaffected by the choice of η ; and η may be chosen to adjust the response for some other input of interest, though a value of one half is generally appropriate.

Higher order problems should first be "simplified" by a partial fraction expansion, though the initial conditions should be incorporated before the expansion. The approach lends itself to parallel processing.

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